# Harmonic Expectation of Harmonic Mean of Random Variables

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**Abstract -** An interesting property of harmonic expectation, whose concept had been introduced on the basis of harmonic mean, and which was consequently defined mathematically, has been identified in the current attempt. The property describes an interesting result on harmonic expectation of harmonic mean of random variables. Description of the property has been presented in this article.

**Keywords:** Random Variable, Harmonic expectation, Harmonic Mean, Property.

## 1. INTRODUCTION

Average [1 , 12] is a single entity that represents a list/set/class of many entities (i.e. many entities) while expectation (of a random variable) is the theoretical average of the possible values assumed by the variable and it is mathematically defined as the weighted average of its all possible values assumed by the random variable with their respective probabilities as the corresponding weights [3 , 5 , 13 , 14 , 16 , 21] . At first, expectation was mathematically defined on the basis of arithmetic mean [2 , 4 , 17] and accordingly defined as the weighted arithmetic mean of its all possible probabilities as the corresponding weights. It was then termed as mathematical expectation [3, 5 , 13 , 14 , 16 , 21] values with their respective]. Later on in another study, it was termed as arithmetic expectation [5 , 10]. Moreover, three more concepts of expectation namely geometric expectation [5 , 6 , 10], harmonic expectation [5 , 7 , 8, 10] and quadratic expectation [9 , 10 , 11] had been introduced and defined mathematically using the concepts of geometric mean [2 , 4 , 18], harmonic mean [2 , 4 , 19] and quadratic mean [15 , 20] respectively. Study had also been done of the properties of these four concepts of expectation and some properties were identified in those attempts [3 , 6 , 7 , 8 , 9 , 11 , 13 , 14 , 16 , 21]. An interesting property of harmonic expectation has been identified in the current attempt. The property describes an interesting result on harmonic expectation of harmonic mean of random variables. Description of the property has been presented in this article.

## 2. HARMONIC EXPECTATION OF A RANDOM VARIABLE

Let us consider a random variable denoted by X.

If the random variable X is discrete and assumes the non-zero real values

$$x_1 , x_2 , \ldots\ldots , x_N$$

with respective probabilities

$$p_1 , p_2 , \ldots\ldots , p_N$$

then the harmonic expectation of X, denoted by $E_H(X)$, is defined by

$$E_H (X) = \left( \sum_{i=1}^{N} p_i x_i^{-1} \right)^{-1} \qquad (2.1)$$

If X is a real valued discrete random variable assuming the countable infinite (denumerable) values

$x_1, x_2, \ldots\ldots\ldots$

with respective probabilities

$p_1, p_2, \ldots\ldots\ldots$

then $E_H(X)$ can be defined by

$$E_H(X) = \left(\sum_{i=1}^{\infty} p_i x_i^{-1}\right)^{-1} \quad (2.2)$$

provided $\sum_{i=1}^{\infty} p_i x_i^{-1}$ is convergent and is a finite number.

Again, if X is a continuous and assumes real values in the interval

(a , b) or [a , b) or (a , b] or [a , b]

with either or both of a & b as finite or infinite,

having probability density function f(x),

then $E_H(X)$ can be defined by

$$E_H(X) = \left\{\int_a^b x^{-1} f(x)\, dx\right\}^{-1} \quad (2.3)$$

## 3. HARMONIC EXPECTATION OF FUNCTION OF RANDOM VARIABLE

If $\varphi$(X) is a function of the real valued random variable X then proceeding with the same logic the harmonic expectation of $\varphi$(X), denoted by $E_H\{\varphi(X)\}$, can be defined as follows:

**Definition (3.1):**

If a real valued discrete random variable X assumes the values $x_1, x_2, \ldots\ldots, x_N$ with corresponding probabilities $p_1, p_2, \ldots\ldots, p_N$, then the harmonic expectation of a function $\varphi$(X) of X, denoted by EH{$\varphi$(X)}, can be defined by

$$E_H\{\varphi(X)\} = \left[\sum_{i=1}^{N} p_i\{\phi(x_i)\}^{-1}\right]^{-1} \quad (3.1)$$

**Definition (3.2):**

If X is a real valued discrete random variable assuming countable infinite (denumerable) values $x_1, x_2, \ldots\ldots\ldots$ with corresponding probabilities $p_1, p_2, \ldots\ldots\ldots$, then the harmonic expectation of a function $\varphi$(X) of X, denoted by $E_H\{\varphi(X)\}$, can be defined by

$$E_H\{\varphi(X)\} = \left[\sum_{i=1}^{\infty} p_i\{\phi(x_i)\}^{-1}\right]^{-1} \quad (3.2)$$

provided $\sum_{i=1}^{\infty} p_i\{\phi(x_i)\}^{-1}$ is convergent and is a finite number.

**Definition (3.3):**

If X is a real valued continuous random variable assuming real values in the intervals

(a , b) or [a , b) or (a , b] or [a , b]

with either or both of a & b as finite or infinite,

having probability density function f(x),

the harmonic expectation of a function $\varphi$ (X) of X, denoted by EH{$\varphi$ (X)}, can be defined by

$$E_H\{\varphi(X)\} = \left[\int_a^b \{\phi(x)\}^{-1} f(x)\,dx\right]^{-1} \qquad (3.3)$$

## 4. RELATION BETWEEN HARMONIC AND ARITHMETIC EXPECTATIONS

Arithmetic expectation [5 , 10] the real valued random variable X, denoted by EA(X), is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i$$

when X is discrete and finite

and by

$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i$$

provided $\sum_{i=1}^{\infty} p_i x_i$ is convergent and is a finite number,

when X is discrete and countable infinite (denumerable)

and also by

$$E_A(X) = \int_a^b x.f(x)dx$$

when X is continuous.

Similarly, the arithmetic expectation of a function $\varphi$ (X) of X, denoted by $E_A\{\varphi(X)\}$, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i x_i$$

when X is discrete and finite

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i \phi(x_i)$$

provided $\sum_{i=1}^{\infty} p_i\phi(x_i)$ is convergent and is a finite number,

when X is discrete and countable infinite (denumerable)

and also by

$$E_A\{\varphi(X)\} = \int_a^b \{\phi(x)\} f(x)\, d^x$$

when X is continuous.

Putting

$$\varphi(X) = X^{-1}$$

in any of the three definitions of EA$\{\varphi(X)\}$ and in any of the three definitions of $E_H\{\varphi(X)\}$, it is obtained that

$$EH(X) = \{E_A(X-1)\}^{-1} \qquad (4.1)$$

Thus, the harmonic expectation of a random variable X is the reciprocal of the arithmetic expectation of its reciprocal (i.e. of $X^{-1}$).

In a similar manner it can be obtained that

$$E_H\{\varphi(X)\} = E_A[\{\varphi(X)\}^{-1}]^{-1} \qquad (4.2)$$

Thus,

"The harmonic expectation of a random variable X is the reciprocal of the arithmetic expectation of its reciprocal i.e. of $X^{-1}$ and also the harmonic expectation of a function $\varphi(X)$ of a random variable X is the reciprocal of the arithmetic expectation of the reciprocal of $\varphi(X)$ i.e. of $\{\varphi(X)\}-1$."

## 5. HARMONIC EXPECTATION OF HARMONIC MEAN

Let

$$X_1, X_2, \ldots\ldots\ldots, X_k$$

be k real non-zero valued random variables.

Then by equation (4.1),

$$E_H(X^{-1}) = \{E_A(X)\}^{-1}$$

$$\text{or} \quad \{E_H(X^{-1})\}^{-1} = E_A(X)$$

Accordingly,

$$E_H\{(X_1 + X_2 + \ldots\ldots + X_k)^{-1}\} = \{E_A(X_1 + X_2 + \ldots\ldots + X_k\}^{-1}$$

$$\text{or} \quad [E_H\{(X_1 + X_2 + \ldots\ldots + X_k)^{-1}\}]^{-1} = \{EA(X_1 + X_2 + \ldots\ldots + X_k)\}$$

By additive property of arithmetic expectation [3 , 13 , 14 , 16 , 21],

$$E_A(X_1 + X_2 + ....... + X_k) = E_A(X_1) + E_A(X_2) + ....... + E_A(X_k)$$

Therefore,

$$[E_H\{(X_1 + X_2 + ....... + X_k)^{-1}\}]^{-1} = \{E_H(X_1^{-1})\}^{-1} + \{E_H(X_2^{-1})\}^{-1} + ....... + \{E_H(X_k^{-1})\}^{-1} \quad (5.1)$$

This can be regarded as additive property of harmonic expectation which can be sated as follows:

"The reciprocal of the harmonic expectation of the reciprocal of sum of a number of random variables is the sum of the reciprocals of the individual harmonic expectation of the reciprocals of the variables."

Now, from equation (5.1), it is obtained that

$$[E_H\{(\frac{X_1 + X_2 + \cdots ........ + X_k}{k})^{-1}\}]^{-1} = \frac{1}{k}[\{E_H(X_1^{-1})\}^{-1} + \{E_H(X_2^{-1})\}^{-1} + ....... + \{E_H(X_k^{-1})\}^{-1}] \quad (5.2)$$

The term

$$(\frac{X_1 + X_2 + \cdots ........ + X_k}{k})^{-1}$$

is the harmonic mean of

$$X_1^{-1}, X_2^{-1}, .........., X_k^{-1}$$

while the term

$$\frac{1}{k}[\{E_H(X_1^{-1})\}^{-1} + \{E_H(X_2^{-1})\}^{-1} + ....... + \{E_H(X_k^{-1})\}^{-1}]$$

is the harmonic mean of

$$E_H(X_1^{-1}), E_H(X_2^{-1}), ......., E_H(X_k^{-1})$$

Thus this equation implies that the harmonic expectation of the harmonic mean of

$$X_1^{-1}, X_2^{-1}, .........., X_k^{-1}$$

is the harmonic mean of the individual harmonic expectations of

$$X_1^{-1}, X_2^{-1}, ......., X_k^{-1}$$

Accordingly, the harmonic expectation of the harmonic mean of

$$X_1, X_2, .........., X_k$$

is the harmonic mean of the individual harmonic expectations of

$$X_1, X_2, .........., X_k$$

Thus, the following theorem has been obtained:

**Theorem (5.1):**

Harmonic Expectation of Harmonic Mean of a number of random variables is the Harmonic Mean of the individual Harmonic Expectations of the variables i.e. if

$$X_1 , X_2 , \ldots\ldots\ldots , X_k$$

are k random variables then

$$E_H \{\text{Harmonic Mean } (X_1 , X_2 , \ldots\ldots , X_k)\} = \text{Harmonic Mean } \{E_H (X_1) , E_H (X_2) , \ldots\ldots , E_H (X_k)\}$$

**Remark (5.1):**

Similarly, one can obtain that

Harmonic expectation of harmonic mean of

$$\varphi (X_1) , \varphi (X_2) , \ldots\ldots\ldots , \varphi (X_k)$$

is the harmonic mean of the individual harmonic expectation of them i.e.

$$E_H [\text{Harmonic Mean } \{\varphi (X_1) , \varphi (X_2) , \ldots\ldots\ldots , \varphi (X_k)\}]$$
$$= \text{Harmonic Mean } [E_H\{\varphi (X_1)\} , E_H\{\varphi (X_2)\} , \ldots\ldots\ldots , E_H\{\varphi (X_k)\}]$$

## 6. CONCLUSION

Additive property of arithmetic expectation namely

"The Additive Expectation of the sum of a number of random variables is the sum of the individual Additive Expectations of the variables"

implies that

"The Additive Expectation of the Additive Mean of a number of random variables is the Additive Mean of the individual Additive Expectations of the variables"

The property of Quadratic Expectation, derived in an earlier study [11], is similar to this property of Additive Expectation.

Similarly, the property of Harmonic Expectation derived here is also similar to this property of Additive Expectation.

At this stage, it is to be investigated whether geometric expectation carries some property similar to that of arithmetic expectation, quadratic expectation and harmonic expectation.

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